Hamromaster

सुनौलो भविष्य हातमा **Hamromaster** को साथमा

Hamromaster

COMPLETE NOTES

For Hamromaster Online Classes

Whatsapp: 9840842566

Hamromaster

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9840842566

notes@hamromaster.com

www.hamromaster.com

..1.1 : Basic concept.

...mentary logic
...nd $\wedge$ or $\vee$; if ... then, if and only if $\Leftrightarrow$

$p \Rightarrow q$
converse $q \Rightarrow p$
inverse $\sim p \Rightarrow \sim q$
contrapositive $\sim q \Rightarrow \sim p$.

$q \Rightarrow p$ converse
inverse of $\Rightarrow \sim q$.
contr

Negation:
3 is rational number.
3 is not a rational number.
$p \Rightarrow q$
if 3 is a rational number then 5 is a prime number.

De-Morgan's law

Tautology $\rightarrow$ last true first whatever $p \wedge q$
$\sim p \vee \sim q$ (negation p or neg q)
$p \vee q$: $\sim p \wedge \sim q$

$p \Leftrightarrow q = (p \wedge \sim q) \vee (q \wedge \sim p)$

Equivalence : Two statements S_1 & S_2 are said to be equivalent if their corresponding truth values are same i.e $S_1 \Leftrightarrow S_2$ is a tautology & denote by $S_1 \equiv S_2$ iff $S_1 \Leftrightarrow S_2$
e.g : $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$

T

T

T

T

since $[\sim(p \wedge q)] \leftrightarrow [\sim p \vee \sim q]$ is a tauto

so, $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Techniques of proof:

Direct Method:

if n is odd an integer (i.e odd) then n² odd.

pf: Let n is odd number, then we can

$n = 2k+1$

where $k \in \mathbb{Z}$

then $n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2[2k(k+1)] + 1$

Hence k is an integers, so k+1 is also integers then 2k(k+1) also integer.

Hence, $n^2 = 2[2k(k+1)]+1$ so, an odd number.

(2) if n is an even number then n^2 is an ev.
number

$n = 2k$

$n^2 = (2k)^2$

$= 2(\ )$

n^2 is also an integer.

Contradiction:

$\sqrt{2}$ is an irrational number.

$\left\{ \begin{array}{l} Q = \text{rational} = \frac{p}{q}, q \neq 0 \\ \overline{Q} = \text{irrational} \quad p, q \in Z \end{array} \right.$

pr

if possible suppose that $\sqrt{2}$ is a rational number, i.e $\sqrt{2} = \frac{p}{q}$, $q \neq 0$ + $p, q \in Z$

$\Rightarrow 2 = \frac{p^2}{q^2}$

$\Rightarrow p^2 = 2q^2$, Here p^2 + q^2 are both integers. [so p^2 is even number only when p is an even number.

so, $p = 2k$

$4k^2 = 2q^2$

$q^2 = 2k^2$

so, q also even i.e q even.

Mathematical Induction Method

1/9/8

De - Morgan's law

(i) $(A \cup B)^c = A^c \cap B^c$

(ii) $(A \cap B)^c = A^c \cup B^c$

(iii) $A \cup B = \{x : x \in A \text{ or } x \in B\}$ [at least one]

(iv) $A \cap B = \{x : x \in A \text{ and } x \in B\}$ [both/all]

prove that : $B - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (B - A_i)$

proof :

$B - \bigcup_{i=1}^{n} A_i = \{x : x \in B \text{ and } x \notin \cup A_i\}$

$= \{x : x \in B \text{ and } x \notin A_i \text{ for all } i = 1, 2, \ldots\}$

$= \{x : x \in (B - A_i) \text{ for all } i = 1, 2, \ldots n\}$

$= \{x : x \in \cap (B - A_i)\}$

$= \bigcap_{i=1}^{n} (B - A_i)$

Relation :

$R \subseteq A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

ϕ = void relation / null relation.

$A \times B = R$, universal relation.

$A \times A$

$R = \{(1,1), (2,2)\}$ identity relation.

Some special types of relation.

① Reflexive relation : A relation R on A is said to be reflexive if $\forall x \in A : (x, x) \in R$ i.e xRx

e.g :

$A = \{ \text{lines} \}$ $R = \leq$ on $\mathbb{R}$

if l_1 is parallel to l_2

(2) symmetric relation: A relation R on A is said to symmetric $\forall x, y \in A$, $(x, y) \in R \Rightarrow (y, x) \in R$.

(3) Transitive Relation: if $(x, y) \in R$ & $(y, z) \in R \Rightarrow (x, z) \in R$. xRy & $yRz \Rightarrow xRz$.

Equivalence Relation:

R on $\mathbb{N}$, prove that this is an equivalence relation $R = \{(x, y) : x \in \mathbb{N}, y \in \mathbb{N}$ & $x - y$ is divisible by 5$\}$

(i) Reflexive: Let $x \in \mathbb{N}$, then $x - x = 0$, is divisible by 5.

(ii) symmetric: Let $x, y \in \mathbb{N}$ then $x - y$ is divisible by 5.

Now,

$y - x = -(x - y)$ is also divisible by 5.

iii) Transitive: $x, y, z \in \mathbb{N}$ $x - y$ is divisible by 5.

$y - x$ is divisible by 5,

$y - z$ is divisible is by 5.

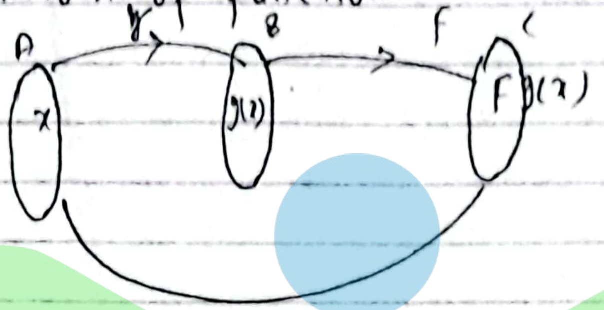
Now,

$x - z = (x-y) + (y-z)$

$x - z$ is divisible by 5.

function.

Composition of function:



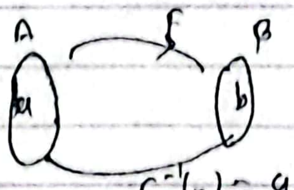
$f \circ g$

Let $g : A \to B$ & $f : B \longrightarrow C$ be two functions. Then we can define a new function from A to C in which each element of set A is associated with unique element of set C.

i.e $x \in A \Rightarrow f(g(x)) \in C$. It is denoted by $f \circ g$ & known as composition of function f and g.

Inverse function:- Let f be a bijective function. The inverse of the function of $b \in B$ to unique element of A. Such that $f(a) = b$. It is denoted by.

$f^{-1} : B \longrightarrow A$ i.e $f^{-1}(b) = a$.



$f^{-1}(b) = a$

Examples: Let $f: R \to R$ be defined by $f(x) = x+1$ & f^{-1}

$\forall y \in R$ there exist at least $x \in R$ such that $f(x) = y$. Since f is one to one & onto. Hence inverse of function exist. Let f^{-1} be the inverse function such that,

$f^{-1}(y) = x$ ... (i)

We have,

$f(x) = x+1$

$y = x+1$

$x = y-1$ ... (ii)

From (i) & (ii) we get,

$f^{-1}(y) = y-1$

$f^{-1}(y) = y-1$ Replacing y by x Hence y is a dummy variables.

$f^{-1}(x) = x-1$

e.g. $f : R \to R$ be a function $f(x) = x^3$, It is bijective or not.

Sol^n,

$f(x_1) = f(x_2)$

$x_1 = x_2$

$x_1^3 - x_2^3 = 0$

$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$x_1 = x_2$ since f is injective.

for onto

$y = x^3$

$x = (y)^{1/3}$ for all y there exists x so

the function is bijective.

$f^{-1}(y) = x$ ...(i)

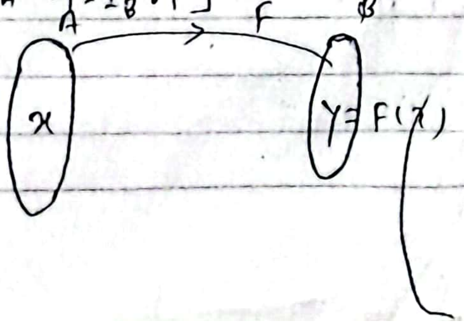
$f^{-1}(y) = y^{1/3}$

Hence y is dummy variables

$f^{-1}(x) = x^{1/3}$.

TH: The composition of any function and identity function is the function itself.

$[f \circ I_A = f = I_B \circ f]$



$f: A \to B$ by $f(x) = y$
$I_A : A \to A$ by $I_A(x) = x$
$I_B : B \to B$ by $I_B(y) = y$.

Let $f: A \to B$ be a function defined by $f(x) = y \ \forall x \in A$ and the identity function $I_A : A \to A$ is defined by $I_A(x) = x, \forall x \in A$ and the identity function $I_B : B \to B$ is defined by $I_B(y) = y \ \forall y \in B$.

Now,

$[f \circ I_A]x = f(I_A(x)) = f(x) = y$ &

$(I_B \circ f)x = I_B(f(x)) = f(x) = y$. Hence proof.

I.V.I

TH: Let $f: x \to y$ be a function. Then f^{-1} exists iff f is bijective [i.e f is 1-1 & onto].

Proof: Let $f: x \to y$ be a function and $f^{-1}: y \to x$ exists. We show that f is 1-1 and onto.

f is 1-1.

$\forall x_1, x_2 \in X$.

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

OR

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

for one-one

let $x_1, x_2 \in X$ &

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow f^{-1}(f(x_1)) = f^{-1}(f(x_2))$ operating f^{-1}

$\Rightarrow (f^{-1} \circ f)(x_1) = (f^{-1} \circ f)x_2$

$\Rightarrow I_X(x_1) = I_X(x_2)$

where I_x is the identity function of x.

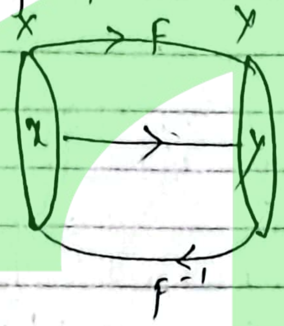
$\Rightarrow x_1 = x_2$

for onto :- Let $y \in Y$ (y is arbitrary)

Since $f^{-1}: Y \to X$ exists.

Then $f^{-1}(y) = x$ i.e $y = f(x)$

$y = f(x)$



Conversely suppose that f is one to one and onto.

Suppose that f is both one-to-one and onto. we show that $f^{-1}: Y \to X$ exists. since f is onto. so $\forall y \in Y$, there exist $x \in X$ such that $f(x) = y$.

further f is one-to-one i.e $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

This shows that for each $y \in Y$, there exists a unique $x \in X$. This defines a new function $g: Y \to X$ such that,

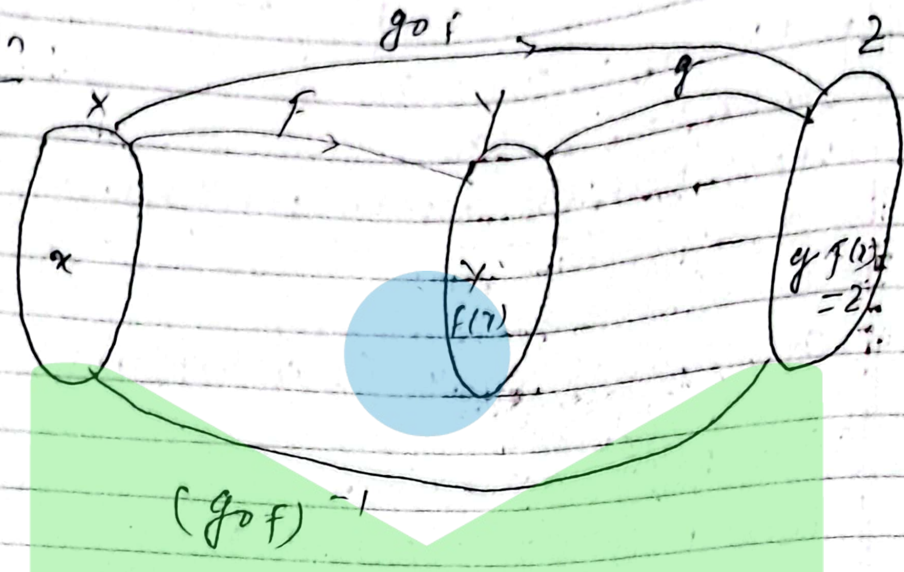
$g(y) = x \Leftrightarrow f(x) = y$, $\forall x \in X$ & $y \in Y$.

This function g is the inverse function of f.

This shows that $f^{-1} = g$ exists.

Let $f: X \to Y$ and $g: Y \to Z$ be two bijective functions. Then the composite function $gof: X \to Z$ is bijective and $(gof)^{-1} = f^{-1} o g^{-1}$

Sol^n:



Proof:

For one-one

Let $x_1, x_2 \in X$ &

$x_1 \neq x_2$

$f(x_1) \neq f(x_2)$ [f is 1-1]

$gf(x_1) \neq g(f(x_2))$ [g is 1-1]

$gof(x_1) \neq gof(x_2)$ [def^n]

$\therefore$ gof is injective (1-1).

For onto surjective

Let $z \in Z$, since g is onto, so $\exists\, y \in Y$ such that $z = g(y)$. As f is onto, so for $y \in Y$, $\exists\, x \to X$ such that $y = f(x)$

Now,

$z = g(y) = g(f(x)) = gof(x)$

Hence, for each $z \in Z$, $\exists\, x \in X$ such that,

$gof(x) = z$

gof is surjective. Therefore this is bijective & it's inverse exists.

To show $(gof)^{-1} = f^{-1}og^{-1}$

for $z \in Z$, $(gof)^{-1}z = x$

$\Leftrightarrow z = gof(x)$

$\Rightarrow z = g(f(x))$

$\Rightarrow g^{-1}(z) = g^{-1}og(f(x))$ operating g^{-1}

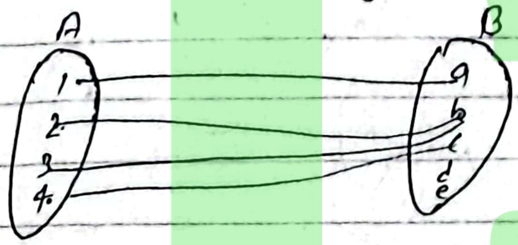
$\Rightarrow g^{-1}(z) = f(x)$

$\Leftrightarrow f^{-1}(g^{-1}(z)) = x$

$\Leftrightarrow (f^{-1}og^{-1})z = x$

Hence $(gof)^{-1} = f^{-1}og^{-1}$.

Inverse Image (pre-image) of an element.



$f^{-1}(a) = \{1\}$

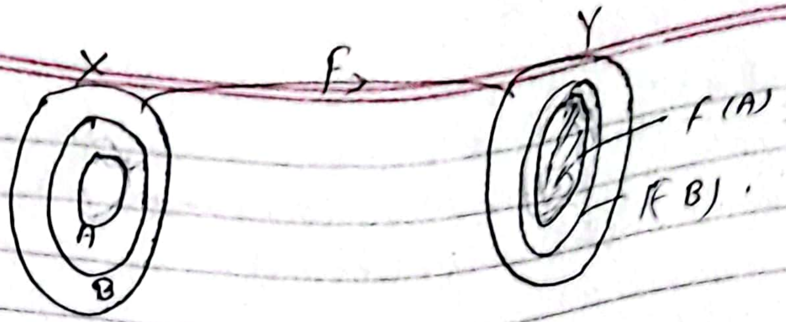
$f^{-1}(b) = \{2, 3\}$

$f^{-1}(c) = \{3\}$

$f^{-1}(d) = \phi, \{\ \}$

THEOREMS ON INVERSE IMAGE.

Th: 1: let $f: X \longrightarrow Y$ be a function. $A, B \in X$ & if $A \subseteq B$ then $f(A) \subseteq f(B)$.



Pf:

Let $y \in f(A)$

$\Rightarrow f^{-1}(y) \in A$

$\Rightarrow f^{-1}(y) \in B$ $[\because A \subseteq B]$

$\Rightarrow y \in f(B)$

Hence,

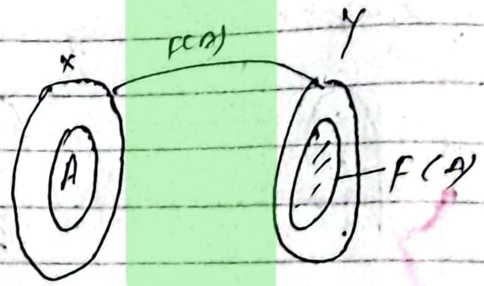
$f(A) \subseteq f(B)$.

TH: 2 Let $f: X \to Y$ be a function, $A \subseteq X$ & $B \subseteq Y$.

prove that,

(a) $A \subseteq f^{-1} f(A)$ &

(b) $f(f^{-1}(B)) \subseteq B$



(a) Let

$x \in A$,

$f(x) \in f(A)$

$\Rightarrow f^{-1} f(x) \in f^{-1} f(A)$

$\Rightarrow x \in f^{-1} f(A)$.

$\therefore A \subseteq f^{-1} f(A)$.

(b) let $y \in f(f^{-1}(B))$ ... *

By def^n of $x \in f^{-1}(B)$ there exists $y \in Y$ s.t. $y = f(x)$

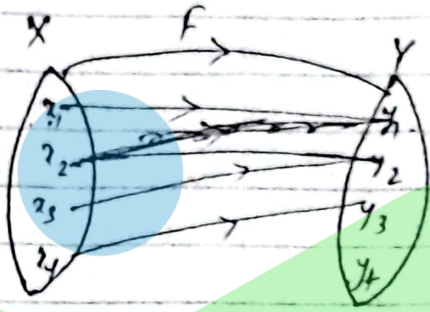
$\Rightarrow f(x) \in B$

$y \in B \ldots$ (**) This can be done for every

$\therefore f(f^{-1}(B)) \subseteq B$.

PH:- Let $f: X \to Y$ be a function, $A \subseteq X$ and $B \subseteq Y$ then equality may not be hold true i.e $A \neq f^{-1}(f(A))$ & $f f^{-1}(B) \neq B$

Sol^{n}:- Let $A = \{x_2, x_3\}$ then $f(A) = \{y_1, y_2\}$



$\Rightarrow f^{-1}(f(A)) = f^{-1}\{y_1, y_2\}$

$\therefore f^{-1} f(A) = \{x_1, x_2, x_3\}$

So, $A \subseteq f^{-1} f(A)$

$\Rightarrow$ Define $f: X \to Y$ such that $f(x_1) = y_1$, $f(x_2) = y_2$, $f(x_3) = y_2$, $f(x_4) = y_3$

Then obviously, $f^{-1} f(A) = f^{-1}(y_1, y_2)$
$= \{x_1, x_2, x_3\}$

Thus, $A \neq f^{-1} f(A)$.

Again, if we take $B = \{y_1, y_2, y_3, y_4\}$ then

$$f f^{-1}(B) = f(x_1, x_2, x_3, x_4)$$
$$= (y_1, y_2, y_3) \neq B.$$

Hence, $f(f^{-1}(B)) \neq B$.

THEOREM: Let $f: X \longrightarrow Y$ be a functions and $A, B \subseteq Y$, then

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Soln,

(i) Let $x \in f^{-1}(A \cup B)$ be an arbitrary element in X. then,

$\Rightarrow f(x) \in A \cup B$.

$\Rightarrow f(x) \in A$ or $f(x) \in B$

$\Rightarrow x \in f^{-1}(A)$ or $x \in f^{-1}B$

$\Rightarrow x \in f^{-1}(A) \cup f^{-1}(B)$

$\therefore f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) Let $x \in f^{-1}(A \cap B)$

$\Rightarrow f(x) \in A \cap B$

$\Rightarrow f(x) \in A$ and $f(x) \in B$

$\Rightarrow x \in f^{-1}(A)$ and $x \in f^{-1}(B)$

$\Rightarrow x \in f^{-1}(A) \cap f^{-1}(B)$

THEOREM: Let $f: X \longrightarrow Y$ be a function. If $A, B \subseteq X$ and $C, D \subseteq Y$, then $f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)$.

proof.

Let, $x \in f^{-1}(C-D) \Leftrightarrow f(x) \in (C-D)$

$\Leftrightarrow f(x) \in C$ and $f(x) \notin D$

$\Leftrightarrow x \in f^{-1}(C)$ and $x \notin f^{-1}(D)$

$\Rightarrow x \in [f^{-1}(C) - f^{-1}(D)]$

$\therefore f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D).$

THEOREM: Let $f: X \rightarrow Y$ be a function. Let A and B be subsets of X.
Prove that $f(A \cap B) \subset f(A) \cap f(B)$.
Also, give an example to show that $f(A \cap B) \neq f(A) \cap f(B)$.

proof:-
Choose $y \in f(A \cap B)$ be arbitrarily then there exists an element
$x \in (A \cap B)$ such that $y = f(x)$
On the other hand, $x \in A \cap B \Rightarrow x \in A$ & $x \in B$
$\Rightarrow f(x) \in f(A)$ and $f(x) \in f(B)$
$\Rightarrow f(x) \in f(A) \cap f(B)$
i.e $y \in f(A) \cap f(B)$
Thus, $y \in f(A \cap B) \Rightarrow y \in f(A) \cap f(B)$
Hence, $f(A \cap B)$ $f(A \cap B) \subset f(A) \cap f(B)$.

another
Let $f: X \rightarrow Y$ be a function defined by
$f(x) = x^2 + 1$.
Suppose, $A = \{1\}$ & $B = \{-1\}$ be the subsets of X. Then clearly $f(1) = f(-1) = 2$.
So, $f(A) = f(B) = \{2\}$ & therefore $f(A) \cap f(B) = \{2\}$. --- (1)
But $A \cap B = \phi$ so, $f(A \cap B) = \phi$.
$\therefore f(A \cap B) \neq f(A) \cap f(B)$.